

EXAMPLE 1 Find a least-squares solution of the inconsistent system $Ax = b$ for

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

✓ rank $[A|b] > \text{rank } A \leadsto$ inconsistent
 ✓ cols of A are independent $\rightarrow$ invertible

$$\begin{aligned} Ax &= b \\ A^T A \hat{x} &= A^T b \\ \hat{x} &= (A^T A)^{-1} A^T b \end{aligned}$$

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \rightarrow \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} [A^T A]^{-1}$$

$$\frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 20 & -2 & 4 \\ -4 & 34 & 16 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \hat{x}$$

EXAMPLE 4 Find a least-squares solution of $Ax = b$ for

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

rank $[A|b] > \text{rank } A$

NOV....

$$\hat{x} = \begin{bmatrix} 32 \\ 4050 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$\textcircled{1} (A^T A)^{-1}: A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 90 \end{bmatrix}$$

$$\textcircled{2} A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 45 \end{bmatrix}$$

diagonal!

$$\begin{aligned} Ax &= b \\ A^T A x &= A^T b \\ \begin{bmatrix} 4 & 0 \\ 0 & 90 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 8 \\ 45 \end{bmatrix} \end{aligned}$$

$$\therefore \begin{bmatrix} 4 & 0 & \vdots & 8 \\ 0 & 90 & \vdots & 45 \end{bmatrix}$$

$$x_1 = \frac{8}{4} = 2$$

$$x_2 = \frac{45}{90} = \frac{1}{2} \neq$$

$$\text{SOL}_1 \quad \hat{x} = (A^T A)^{-1} A^T b$$

$$\text{SOL}_2 \quad \underbrace{A^T A} x = \underbrace{A^T b} \rightarrow \text{augmented matrix}$$

$$\text{SOL}_3 \quad \text{orthogonality} \quad \text{orthogonal}$$

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = x \text{ using proj.} \quad c_1 = \frac{a_1 \cdot b}{a_1 \cdot a_1} = \frac{8}{4} = 2, \quad c_2 = \frac{a_2 \cdot b}{a_2 \cdot a_2} = \frac{45}{90} = \frac{1}{2} \therefore x = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

what is the difference of this solution to Example 1 (Q1)

EXAMPLE 2 Find a least-squares solution of $Ax = b$ for

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

$$\text{rank } [A|b] = 4 > 3 = \text{rank } A$$

↓
inconsistent

dependent columns $\rightarrow$ not invertible $\perp$

$$Ax = b$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{bmatrix} \begin{matrix} \\ \sim R_2 - R_3 \\ \sim R_3 - R_4 \\ \end{matrix}$$

$$= \begin{bmatrix} 6 & 2 & 2 & 2 & 4 \\ 0 & 2 & -2 & 0 & -6 \\ 0 & 0 & 2 & -2 & -4 \\ 2 & 0 & 0 & 2 & 6 \end{bmatrix} \sim R_1 - 3R_4 - R_2 - R_3$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & -6 \\ 0 & 0 & 2 & -2 & -4 \\ 2 & 0 & 0 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 & 6 \\ 0 & 2 & -2 & 0 & -6 \\ 0 & 0 & 2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 - x_4 \\ x_4 - 5 \\ x_4 - 2 \\ x_4 \end{bmatrix}$$

$$x = x_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix}$$

∞ many solutions with the same least square

$$x_1 + x_4 = 3 \rightarrow x_1 = 3 - x_4$$

$$x_2 - x_3 = -3 \rightarrow x_2 = -3 + x_3$$

$$x_3 - x_4 = -2 \rightarrow x_3 = x_4 - 2$$

$$\begin{aligned} x_2 &= -3 + x_3 \\ &= -3 + x_4 - 2 \\ &= x_4 - 5 \end{aligned}$$

14. Let $A = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$. Compute $A\mathbf{u}$ and $A\mathbf{v}$, and compare them with $\mathbf{b}$. Is it possible that at least one of $\mathbf{u}$ or $\mathbf{v}$ could be a least-squares solution of $A\mathbf{x} = \mathbf{b}$? (Answer this without computing a least-squares solution.)

columns of A are linearly independent

if unique solution
↓
if 1 x n matrix then
least square

$$A\mathbf{u} = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} \quad \|A\mathbf{x} - \mathbf{b}\| = \left\| \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} \right\| = \sqrt{24}$$

$$A\mathbf{v} = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix} \quad \|A\mathbf{x} - \mathbf{b}\| = \left\| \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \right\| = \sqrt{24}$$

∴ Not possible!

In Exercises 17 and 18, A is an $m \times n$ matrix and $\mathbf{b}$ is in $\mathbb{R}^m$. Mark each statement True or False. Justify each answer.

17. a. The general least-squares problem is to find an $\mathbf{x}$ that makes $A\mathbf{x}$ as close as possible to $\mathbf{b}$.
- b. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ that satisfies $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the orthogonal projection of $\mathbf{b}$ onto $\text{Col } A$.
- c. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ such that $\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
- d. Any solution of $A^T A\mathbf{x} = A^T \mathbf{b}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$.
- e. If the columns of A are linearly independent, then the equation $A\mathbf{x} = \mathbf{b}$ has exactly one least-squares solution.

18. a. If $\mathbf{b}$ is in the column space of A , then every solution of $A\mathbf{x} = \mathbf{b}$ is a least-squares solution. T
- b. The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the column space of A closest to $\mathbf{b}$. F
- c. A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a list of weights that, when applied to the columns of A , produces the orthogonal projection of $\mathbf{b}$ onto $\text{Col } A$. T
- d. If $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$, then $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$. F
- e. The normal equations always provide a reliable method for computing least-squares solutions. F
- f. If A has a QR factorization, say $A = QR$, then the best way to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ is to compute $\hat{\mathbf{x}} = R^{-1} Q^T \mathbf{b}$. F

$A\hat{\mathbf{x}} = \mathbf{b}$ consistent

a) T definition!

b) T $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$

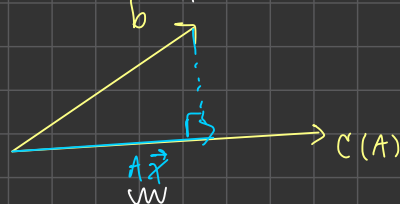
c) F $\|\mathbf{b} - A\mathbf{x}\| \geq \|\mathbf{b} - A\hat{\mathbf{x}}\|$

d) T min theorem

e.) T unique solution

a) T $\|A\hat{\mathbf{x}} - \mathbf{b}\| = 0 \dots$ WTF!?

b) F $A\hat{\mathbf{x}}$ not $\hat{\mathbf{x}}$



c) T

d) F, columns of A should be linearly independent too. Not all $A^T A$ is invertible

e.) F QR algorithm cols A linearly independent.

f.) F $R\mathbf{x} = Q^T \mathbf{b}$ then solve + back substitute

19. Let A be an $m \times n$ matrix. Use the steps below to show that a vector $\mathbf{x}$ in $\mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$ if and only if $A^T A\mathbf{x} = \mathbf{0}$. This will show that $\text{Nul } A = \text{Nul } A^T A$.

a. Show that if $A\mathbf{x} = \mathbf{0}$, then $A^T A\mathbf{x} = \mathbf{0}$.

b. Suppose $A^T A\mathbf{x} = \mathbf{0}$. Explain why $\mathbf{x}^T A^T A\mathbf{x} = 0$, and use this to show that $A\mathbf{x} = \mathbf{0}$.

a) ทั่วไป ($\Rightarrow$)

$$\begin{aligned} A\mathbf{x} &= \mathbf{0} \\ A^T(A\mathbf{x}) &= A^T(\mathbf{0}) \\ A^T A\mathbf{x} &= \mathbf{0} \end{aligned}$$

$$\therefore N(A) = N(A^T A)$$

b) ทั่วไป ($\Leftarrow$)

$$\begin{aligned} A^T A\mathbf{x} &= \mathbf{0} \\ \mathbf{x}^T A^T A\mathbf{x} &= \mathbf{x}^T \mathbf{0} \\ \mathbf{x}^T A^T A\mathbf{x} &= 0 \\ (A\mathbf{x})^T A\mathbf{x} &= 0 \\ \|A\mathbf{x}\|^2 &= 0 \\ \therefore A\mathbf{x} &= \mathbf{0} \end{aligned}$$

20. Let A be an $m \times n$ matrix such that $A^T A$ is invertible. Show that the columns of A are linearly independent. [Careful: You may not assume that A is invertible; it may not even be square.]

$$\begin{aligned} A^T(A\mathbf{x}) &= A^T(\mathbf{0}) \\ A^T A\mathbf{x} &= \mathbf{0} \\ (A^T A)^{-1}(A^T A)\mathbf{x} &= (A^T A)^{-1}\mathbf{0} \\ \mathbf{x} &= \mathbf{0} \end{aligned}$$

A is linearly independent
 $A\mathbf{x} = \mathbf{0}$ only has trivial solution.

suppose $A\mathbf{x} = \mathbf{0}$:

$\therefore A\mathbf{x} = \mathbf{0}$ only has trivial solution.
 $\therefore$ columns of A are linearly independent.

21. Let A be an $m \times n$ matrix whose columns are linearly independent. [Careful: A need not be square.]

- Use Exercise 19 to show that $A^T A$ is an invertible matrix.
- Explain why A must have at least as many rows as columns.
- Determine the rank of A .

a) A has linearly independent columns

$$\begin{aligned} A\mathbf{x} = \mathbf{0} \text{ only has trivial solution} &\rightarrow N(A) = \{\mathbf{0}\} \\ \text{from 19)} \quad N(A) = N(A^T A) &= \{\mathbf{0}\} \\ \therefore \text{columns of } A^T A &\text{ are linearly independent.} \end{aligned}$$

$\downarrow$
 invertible
 square matrix

b) Why A must have $m \geq n$?

since columns of A must span $\mathbb{R}^m \rightarrow$ pivot in row $\rightarrow$ unique sol

idea: $n > m$ then columns can not be linearly independent
 since $\text{span } \mathbb{R}^m$ for m vector
 ถ้ามีมากกว่า m ก็เป็นอิสระ

c) since all columns of A are linearly indep. $\rightarrow$ # basis vector of $C(A)$
 $= n = \text{rank}$

Q7: Applications to linear models: Linear regression least squares

Ref: <https://calcworkshop.com/linear-regression/least-squares-regression-line/>

You can use a programming language to find the answer for this question.

Given the following table of measured height and weight of 7 individuals,

Dataset

Height (in)	Weight (lb)
48	93
50	105
53	102
55	118
61	121
64	135
73	180

$$y = mx + d$$

$$\begin{bmatrix} 1 & 48 \\ 1 & 50 \\ 1 & 53 \\ 1 & 55 \\ 1 & 61 \\ 1 & 64 \\ 1 & 73 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 93 \\ 105 \\ 102 \\ 118 \\ 121 \\ 135 \\ 180 \end{bmatrix}$$

- a) Assume that the weight (w) and height (h) are related by the following equation:

$$w = \beta_0 + \beta_1 h$$

find the unknown variable β_0, β_1 .

- b) Calculate and plot the residuals using the β found

a)

$$\begin{bmatrix} 1 & 48 \\ 1 & 50 \\ 1 & 53 \\ 1 & 55 \\ 1 & 61 \\ 1 & 64 \\ 1 & 73 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 93 \\ 105 \\ 102 \\ 118 \\ 121 \\ 135 \\ 180 \end{bmatrix}$$

constant b

β_0

β_1

x, x^1

x

independent rank $[A|b] > \text{rank } A$

$$A x = b$$

$$A^T A x = A^T b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 48 & 50 & 53 & 55 & 61 & 64 & 73 \end{bmatrix} \begin{bmatrix} 1 & 48 \\ 1 & 50 \\ 1 & 53 \\ 1 & 55 \\ 1 & 61 \\ 1 & 64 \\ 1 & 73 \end{bmatrix} = \begin{bmatrix} 1 & 404 \\ 404 & 23784 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 48 & 50 & 53 & 55 & 61 & 64 & 73 \end{bmatrix} \begin{bmatrix} 93 \\ 105 \\ 102 \\ 118 \\ 121 \\ 135 \\ 180 \end{bmatrix} = \begin{bmatrix} 854 \\ 50771 \end{bmatrix}$$

Augmented

$$\begin{bmatrix} 1 & 404 & 854 \\ 404 & 23784 & 50771 \end{bmatrix} \sim R_2 - 404R_1 \rightarrow \begin{bmatrix} 1 & 404 & 854 \\ 0 & -139432 & -294245 \end{bmatrix}$$

$\beta_0 \approx 2.11$

$\beta_1 \approx 1.56$

$\beta_1 + 404 \beta_0 = 854$

$\beta_1 \approx 854 - 404(2.11)$

Q8) Least squares fit of other curves

You can use a programming language to find the answer for this question.

Given that y is generated by the following equation:

$$y_{\text{clean}} = \beta_0 + \beta_1 x + \beta_2 x^2$$

$$y_{\text{noisy}} = y_{\text{clean}} + n$$

for $x = -3, -2, -1, 0, 1, 2, 3$ and n is some random noise with mean 0, we recorded the following value pairs:

$x = [-3, -2, -1, 0, 1, 2, 3]$
 $y_{\text{clean}} = [4.3000, 2.6000, 1.5000, 1.0000, 1.1000, 1.8000, 3.1000];$
 $n = [-0.0830, 0.2203, -0.4999, -0.1977, -0.3532, -0.4077, -0.3137];$
 $y_{\text{noisy}} = [4.2170, 2.8203, 1.0001, 0.8023, 0.7468, 1.3923, 2.7863];$

Answer the followings:

- find the parameters β_{clean} using (x, y_{clean}) and β_{noisy} using (x, y_{noisy}) ;
- why is the residual error using β_{clean} with the equation $y = \beta_0 + \beta_1 x + \beta_2 x^2$ while it is not zero when using β_{noisy} ?

===== end of Tut 7 =====

$$A) \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_7 & x_7^2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} y_{\text{clean}} \end{bmatrix}$$

$$\text{using } X^T X, X^T y \rightarrow \text{inter least square for } \beta_{\text{clean}} = \begin{bmatrix} -61.1088 \\ 3.1727 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_7 & x_7^2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} y_{\text{noisy}} \end{bmatrix}$$

$$\text{using } X^T X, X^T y \rightarrow \text{for } \beta_{\text{noisy}}$$

- Why is the residual error using β_{clean} with the equation $y = \beta_0 + \beta_1 x + \beta_2 x^2$ zero while it is not zero when using β_{noisy} ?

$y_{\text{clean}} = X\beta_{\text{clean}}$ gives the exact solution to $y = \beta_0 + \beta_1 x + \beta_2 x^2$.

On the other hand, using β_{noisy} generates y_{noisy} which has residual error due to the presence of n which is not accounted for in $y = \beta_0 + \beta_1 x + \beta_2 x^2$.

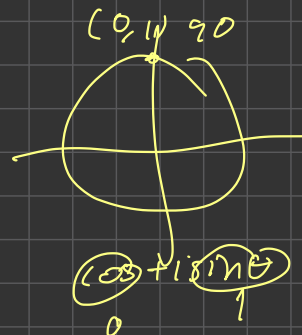
e) Given $x(n) = 3 \sin(\frac{2\pi}{16}n + \frac{\pi}{3})$, what is $X(k)$? How is this answer differ to Q6d.

synthesis

$$x(n) = \frac{1}{N} W^H X(k)$$

$$\begin{aligned} e^{j\theta} - e^{-j\theta} &= 2j \sin \theta \\ \sin \theta &= \frac{e^{j\theta} - e^{-j\theta}}{2j} \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{\pi}{4}} & e^{-j\frac{\pi}{2}} & e^{-j\frac{3\pi}{4}} \\ 1 & e^{-j\frac{\pi}{2}} & e^{-j\pi} & e^{-j\frac{5\pi}{4}} \\ 1 & e^{-j\frac{3\pi}{4}} & e^{-j\frac{3\pi}{2}} & e^{-j\frac{7\pi}{4}} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}$$



$$3 \sin(\frac{2\pi}{16}n + \frac{\pi}{3}) = 3 \left(\frac{e^{j\theta} - e^{-j\theta}}{2j} \right)$$

$$= -\frac{3j}{2} (e^{j\theta} - e^{-j\theta})$$

$$= -\frac{3}{2} e^{j\frac{\pi}{2}} (e^{j(\frac{2\pi}{16}n + \frac{\pi}{3})} - e^{-j(\frac{2\pi}{16}n + \frac{\pi}{3})})$$

$$= -\frac{3}{2} \cdot \underbrace{e^{j\frac{\pi}{2}} \cdot e^{j\frac{\pi}{2}}}_{110 \} k=1} (e^{j(\frac{2\pi}{16}n)} + e^{-j(\frac{2\pi}{16}n)})$$

$$\frac{1}{16}$$

$$\begin{aligned} &e^{j(-\frac{2\pi}{16}n)} \\ &e^{j(\frac{30\pi}{16}n)} \\ &k=15 \end{aligned}$$

$$= \begin{bmatrix} 0 \\ -\frac{24}{2} e^{j\frac{5\pi}{6}} \\ 0 \\ 0 \\ \vdots \\ -\frac{24}{2} e^{j\frac{5\pi}{6}} \end{bmatrix}$$

$$\frac{2\pi}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{4}} & e^{j\frac{4\pi}{4}} & e^{j\frac{6\pi}{4}} \\ 1 & e^{j\frac{4\pi}{4}} & e^{j\frac{8\pi}{4}} & e^{j\frac{12\pi}{4}} \\ 1 & e^{j\frac{6\pi}{4}} & e^{j\frac{12\pi}{4}} & e^{j\frac{18\pi}{4}} \end{bmatrix}$$

Q1) Evaluate $z = (2 - i)^2 + \frac{7-4i}{2+i} + \frac{1}{1+i}$, and give the result in rectangular, trigonometric polar and complex exponential form.

$$z = (2-i)^2 + \frac{7-4i}{2+i} + \frac{1}{1+i}$$

$$= 4 - (2)(2)(i) + i^2 + \frac{7-4i}{2+i} \times \frac{(2-i)}{(2-i)} + \frac{1}{i+1} \times \frac{(i-1)}{(i-1)}$$

$$= 4 - 4i - 1 + \frac{14 - 7i - 8i - 4}{4 + i} + \frac{i - 1}{-2}$$

$$= 3 - 4i + \frac{10 - 15i}{5} - \frac{i - 1}{2}$$

$$= 3 - 4i + 2 - 3i - \frac{i}{2} + \frac{1}{2} = \frac{11}{2} - \frac{15}{2}i$$

① rectangular $\frac{11}{2} - \frac{15}{2}i \rightarrow \tan \theta = -\frac{15}{11} \rightarrow \theta = \tan^{-1}\left(-\frac{15}{11}\right)$, $r = \sqrt{a^2 + b^2} \approx 9.307$

② trig. polar $\text{cis } \theta = 9.301 \text{ cis } \left(\tan^{-1} \left(\frac{-15}{11} \right) \right) \approx -0.938 \text{ rad}$

③ complex expo $1.301 e^{i(\tan^{-1}(-\frac{15}{11}))}$

Q2) Given $y = -2x^2 + 3x - c$, where c is a real number, answer the following questions:

- Find c for the case when there is only 1 repeated root. 1 ΑΙΜΟΥ
- For $c = +2$, find the roots of the equation
- When $c \rightarrow +\infty$, there will only be (repeated/complex/real)? Roots
- When $c \rightarrow -\infty$, there will only be (repeated/complex/real)? Roots

$$a) \quad b^2 - 4ac = 0$$

$$3^2 - (4)(-2)(0) = 0$$

$$q - 8c = 0$$

$$c = \frac{9}{8} \checkmark$$

answer : $\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$
 $= \frac{-3 \pm \sqrt{9 - 8C}}{-4}$

$$b) \quad c = +2 : \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - (4)(-2)(-2)}}{2(-2)} = \frac{-3 \pm \sqrt{-7}}{-4}$$

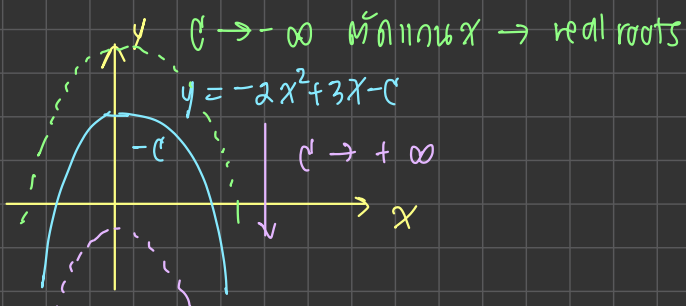
$$x = \frac{-3 \pm \sqrt{9-8C}}{-4} \quad \begin{matrix} \oplus \\ \ominus \end{matrix} \quad = \frac{3}{4} \pm \frac{\sqrt{7}i}{4}$$

c) $C \rightarrow \infty$: there will be complex roots w/h:

$b^2 \in 4a(\theta_c^\dagger)$

d) $c \rightarrow -\infty$: there will be **real roots** why:

$$b^2 = 4 \underbrace{a_w(e_w^*)}_{\text{more positive}} \oplus \sum_i c_i m_i^2 +$$



Q3) Evaluate z^4 given $z = \overline{(1+2i)}$

$$z = \overline{(1+2i)} = 1-2i \rightarrow r = \sqrt{1^2+2^2} = \sqrt{5}$$

$\theta = \tan^{-1}\left(-\frac{2}{1}\right)$
 $\theta \approx -1.107$ rad

De Moivre's theorem

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta)$$

$$(re^{j\theta})^n = r^n e^{jn\theta}$$

principle argument

$$-\pi < \theta \leq \pi$$

$$z \approx \sqrt{5} e^{-j1.107}$$

$$z^4 \approx (\sqrt{5})^4 e^{-j4.429} = 25 e^{-j4.429} = 25 e^{j1.856}$$

Q4) Find the values of a, b for $(a-bi)^2(3+5i) = \overline{(-6-24i)}$.

$$(a-bi)^2(3+5i) = (-6+24i) \rightarrow \text{หาค่าของ } a, b$$

$$(a-bi)^2 = \frac{(-6+24i)}{(3+5i)} \times \frac{(3-5i)}{(3-5i)}$$

$$a^2 - 2abi - b^2 = \frac{-18 + 30i + 72i + 120}{9 + 25}$$

$$a^2 - 2abi - b^2 = \frac{102 + 102i}{34}$$

$$(a^2 - b^2) + i(-2ab) = 3 + 3i$$

$$\text{เทียบส่วน: } a^2 - b^2 = 3 \rightarrow \left(-\frac{3}{2b}\right)^2 - b^2 = 3$$

$$-2ab = 3$$

$$a = -\frac{3}{2b}$$

$$-\frac{9}{4b^2} - b^2 = 3$$

$$-9 - 4b^4 = 12b^2$$

$$-4k^2 - 12k - 9 = 0$$

$$4k^2 + 12k + 9 = 0$$

$$(2k+3)^2 = 0$$

$$k = -\frac{3}{2}$$

$$\therefore b^2 = -\frac{3}{2} \rightarrow b = \sqrt{\frac{3}{2}} i$$

$$a = -\frac{3}{2b} = -\frac{3}{2} \times \sqrt{\frac{2}{3}} \cdot \frac{1}{i} = -\sqrt{\frac{3}{2}} i$$

$$\therefore a = -\sqrt{\frac{3}{2}} i$$

หาค่าของ a, b
แล้ว

$$(a, b) = \left(-\sqrt{\frac{3}{2}} i, \sqrt{\frac{3}{2}} i\right), \left(\sqrt{\frac{3}{2}} i, -\sqrt{\frac{3}{2}} i\right)$$

Solution 2 $\sqrt[n]{\text{root}}$

$$(a+bi)^2 = 3+3i \quad r = \sqrt{3^2+3^2} = \sqrt{18} = 3\sqrt{2}$$

$$\tan \theta = 1 \quad \theta = \frac{\pi}{4}$$

$$a+bi = \sqrt{3+3i} = (\sqrt{18} \operatorname{cis} \frac{\pi}{4})^{\frac{1}{2}} = (\sqrt{18})^{\frac{1}{2}} \left(\operatorname{cis} \frac{\pi}{8}\right)$$

roots $\sqrt[4]{18} \operatorname{cis} \frac{\pi}{8}$, $\sqrt[4]{18} \operatorname{cis} \frac{9\pi}{8}$
 $k=0$, $k=1$
 $+ \frac{2\pi}{2}$

$$a = \sqrt[4]{18} \cos \frac{\pi}{8}, \sqrt[4]{18} \cos \frac{9\pi}{8}$$

$$b = \sqrt[4]{18} \sin \frac{\pi}{8}, \sqrt[4]{18} \sin \frac{9\pi}{8}$$

De Moivre's extended (n roots)

$$z = a + bi = re^{j\theta}$$

$$\omega_k = z^{\frac{1}{n}} = (re^{j\theta})^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j\left(\frac{\theta+2\pi k}{n}\right)}$$

$$= r^{\frac{1}{n}} \left(\operatorname{cis} \left(\frac{\theta+2\pi k}{n}\right)\right)$$

$$(k=0, 1, \dots, n-1)$$

Q5)

 $X(w)$

- a) Given $x = [2, 0, 0, 3]^T$, find DFT coefficients X with $N = 4$.
- b) Given $X = N \cdot [0, 3e^{-j\frac{\pi}{4}}, 0, 0]^T$, find x via IDFT with $N = 4$, and suggest the equation describing $x[n]$ by inspection.
- c) Given $X = N \cdot [2, 3e^{-j\frac{\pi}{4}}, 0, 0]^T$, find solution of x found in Q5b, suggest the equation describing $x[n]$ by inspection.

NOTE:

x sample of function f taken at discrete N time-points

X DFT coefficients
tell us how much of the signal can be expressed using that frequency

w represent different frequencies in each time points

$$x = \sum |X(w)| (\cos(\omega t + \angle(X(w)))) \leftarrow f$$

a) DFT analysis

$$\begin{aligned}
 \overset{\text{freq } k}{X(w)} &= \begin{bmatrix} w^{kn} \\ \vdots \end{bmatrix} \overset{\text{time } n}{x(n)} \\
 \theta = \frac{2\pi kn}{N} &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{\pi}{2}} & e^{-j\pi} & e^{-j\frac{3\pi}{2}} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} \\ 1 & e^{-j\frac{3\pi}{2}} & e^{-j\frac{6\pi}{2}} & e^{-j\frac{9\pi}{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ e^{-j\frac{3\pi}{2}} \\ e^{-j\frac{12\pi}{4}} \\ e^{-j\frac{18\pi}{4}} \end{bmatrix} \\
 &= \begin{bmatrix} 2 + 3 \\ 2 + 3e^{-j\frac{3\pi}{2}} \\ 2 + 3e^{-j3\pi} \\ 2 + 3e^{-j\frac{9\pi}{2}} \end{bmatrix} \quad \text{Weights}
 \end{aligned}$$

b) DFT synthesis

$$\begin{aligned}
 x(n) &= \frac{1}{N} [w^{-kn}] X(w) \\
 &= \frac{1}{N} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{\pi}{2}} & e^{j\pi} & e^{j\frac{3\pi}{2}} \\ 1 & e^{j\pi} & e^{j2\pi} & e^{j3\pi} \\ 1 & e^{j\frac{3\pi}{2}} & e^{j\frac{6\pi}{2}} & e^{j\frac{9\pi}{2}} \end{bmatrix} \cdot N \cdot \begin{bmatrix} 0 \\ 3e^{-j\frac{\pi}{4}} \\ 0 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} 3e^{-j\frac{\pi}{4}} \\ 3e^{j(\frac{\pi}{2}-\frac{\pi}{4})} \\ 3e^{j(\pi-\frac{\pi}{4})} \\ 3e^{j(\frac{3\pi}{2}-\frac{\pi}{4})} \end{bmatrix} = 3 \begin{bmatrix} e^{-j\frac{\pi}{4}} \\ e^{j\frac{\pi}{4}} \\ e^{j\frac{3\pi}{4}} \\ e^{j\frac{5\pi}{4}} \end{bmatrix} \quad n=1 \\
 &= \begin{bmatrix} 2.12 - 2.12j \\ 2.12 + 2.12j \\ -2.12 + 2.12j \\ -2.12 - 2.12j \end{bmatrix} \quad \text{2nd column only!}
 \end{aligned}$$

c) DFT synthesis

$$\begin{aligned}
 x(n) &= \frac{1}{N} [w^{-kn}] X(w) \\
 &= \frac{1}{N} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{\pi}{2}} & e^{j\pi} & e^{j\frac{3\pi}{2}} \\ 1 & e^{j\pi} & e^{j2\pi} & e^{j3\pi} \\ 1 & e^{j\frac{3\pi}{2}} & e^{j\frac{6\pi}{2}} & e^{j\frac{9\pi}{2}} \end{bmatrix} \cdot N \cdot \begin{bmatrix} 2 \\ 3e^{-j\frac{\pi}{4}} \\ 0 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} 3e^{-j\frac{\pi}{4}} + 2 \\ 3e^{j\frac{\pi}{4}} + 2 \\ 3e^{j\frac{3\pi}{4}} + 2 \\ 3e^{j\frac{5\pi}{4}} + 2 \end{bmatrix}
 \end{aligned}$$

Chapter 8 : eigen

Q1 Lay5e/Ch5.1/pg273/Ex6+7

6. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue. λ

7. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector. x

6)

$$Ax = \lambda x$$

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ eigenvalue $\lambda = -2$

7)

$$Ax = \lambda x$$

$$Ax = 4x$$

nullspace of $A - \lambda I$

$$[A - 4I]x = 0 \rightarrow \text{matrix } A - 4I \text{ consistent?}$$

$$A - 4I = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$$

dependent columns

$$a_1 + a_2 = a_3$$

nontrivial sol ival!

พอใช้!

consider homogeneous augmented matrix

$$\begin{bmatrix} +1 & 0 & +1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix} \xrightarrow{\sim R_3 + R_2 + R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \xrightarrow{\sim R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

↓

$$\therefore x_1 + x_3 = 0$$

$$x_3 = -x_1$$

$$x_2 + x_3 = 0$$

$$x_2 = -x_3$$

$$= x_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis of eigen space

$$x = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \text{ e.g. } \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \text{ with } x_3 = 1$$

$$\text{ดังนั้น } Ax = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix} = \lambda x$$

Q2) Lay5e/Ch5.1/pg274/

Find the eigenvector corresponding to eigenvalue = -2, for the following matrix

14. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}, \lambda = -2$

~~$x_1 = x_3$~~
 $x = x_3 \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$

$$\begin{aligned} A x &= \lambda x \\ A x - \lambda x &= 0 \\ (A - \lambda I) x &= 0 \\ \Rightarrow \begin{bmatrix} 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{bmatrix} &\sim R_3 - R_2 - R_1 \\ \Rightarrow \begin{bmatrix} 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -12 & 4 & 0 \end{bmatrix} &\sim R_1 - 3R_2 \\ \Rightarrow \begin{bmatrix} 0 & 3 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -12 & 4 & 0 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{1}{3} x_3 \\ x_1 - \frac{13}{4} x_2 + \frac{3}{4} x_3 &= 0 \\ x_1 &= \frac{13}{12} x_3 - \frac{3}{12} x_3 \\ x_1 &= \frac{1}{3} x_3 \end{aligned}$$

$$\therefore x = x_3 \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_2 &= 0 \rightarrow x_1 = x_2 = \frac{x_3}{3} \\ 3x_2 - x_3 &= 0 \rightarrow x_2 = \frac{x_3}{3} \end{aligned}$$

8.13 quiz

Q3) Lay5e/Ch5/pg 274/Q20

20. Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. Justify your answer.

all linearly dependent

① square, ② all linearly dependent columns

$$\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5(x_1 + x_2 + x_3) \\ 5(x_1 + x_2 + x_3) \\ 5(x_1 + x_2 + x_3) \end{bmatrix}$$

$$\text{if } x_1 = x_2 = x_3 = 5 \begin{bmatrix} 3K \\ 3K \\ 3K \end{bmatrix} = 15 \begin{bmatrix} K \\ K \\ K \end{bmatrix}$$

one eigen λ value ~ 15 wrong

by IMT

$Ax = 0$ has non-trivial solution
 $(A - \lambda I)x = 0$ where $\lambda = 0$

$$Ax = \lambda x = 0$$

can find 2 linearly indep eigenvectors?

$$\begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5(x_1 + x_2 + x_3) \\ 5(x_1 + x_2 + x_3) \\ 5(x_1 + x_2 + x_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. Justify your answer.

- Columns of A are linearly dependent.
- $Ax = (A - 0I)x = 0$ has a non-trivial solution $\Rightarrow 0$ is an eigenvalue of A .
- Since the columns of A are identical, any nonzero vector in $\mathbb{R}^3$ whose entries sum to 0 will be an eigenvector corresponding to $\lambda = 0$.